Qualifying Examination in Discrete Mathematics
for the Ph. D. Program February 2006

Let $|S|$ be the cardinality of a set S, N be the set of all positive integers, and a cycle be a 2-regular connected graph.

1. State and prove Hall's Theorem. (15%)

2. Prove that every $2k$-regular graph with even order $n\geq 3$ and $k>0$ has a spanning subgraph with each component being a cycle. (15%)

3. Let S be the set of permutations (a,b,c,d,e,f) on $\{1,2,3,4,5,6\}$ with $a\neq 1,2,3$, $b\neq 2,3$, $c\neq 3$, $d\neq 4,5$, and $e\neq 5$. Find the number $|S|$. (15%)

4. Suppose G is a strongly connected tournament with order $n\geq 3$. Prove that G contains directed cycles of length between 3 and n.(15%)

5. True or False. (If the statement is true, prove it; if it is false, give a counterexample) (10%×4)
   (a) If G is a planar graph with the girth at least 4 then G has a vertex of degree at most 3.

   (b) Let A_n be a set for $n \in N$. If, for each finite set $I\subseteq N$, $|I|\leq|\cup_{n\in I}A_n|$ then there exist $x_n\in A_n$ for all n satisfying $x_a\neq x_b$ for $a\neq b$.

   (c) If a graph G is bipartite then the chromatic index of G is equal to the maximum degree of G.

   (d) If the chromatic number of a graph G is $k\geq 4$ then G has a vertex with degree at least k.