Differential Equations

Ph.D. Qualifying Examination

National Sun Yat-sen University

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Answer all the five problems, each of which carries 20%. The total is 100%. Unless otherwise stated, $x \in \mathbb{R}^n$, Ω is an open subset of $\mathbb{R}^n$ and $H^1(\Omega) = W^{1,2}(\Omega)$ is a Sobolev space.

1. (a) Show that the equation $u_{xx} + u_{yy} = 0$, after the transformation $x = r\cos\theta,\ y = r\sin\theta$, becomes

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

   (b) Show that all harmonic functions satisfy the mean value property. (Hint: May use the identity: $\int_B \Delta u\, dx = \int_{\partial B} \frac{\partial u}{\partial n} dS$.)

2. (a) Let $u(x) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies

$$Lu := \Delta u + \sum_{i=1}^{n} b_i(x) D_i u + c(x)u < 0\ ,$$

   where $c(x) \le 0$ in Ω. Show that if u has a nonpositive minimum value at $x_0 \in \overline{\Omega}$, then $x_0 \in \partial\Omega$.

   (b) Show that the positive solution in $C^2(\Omega) \cap C^0(\overline{\Omega})$ of the boundary value problem ($p > 1$)

$$\Delta u - u^p = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega,$$

   if exists, is unique.

3. Assume that Ω is bounded and connected, and $f \in L^2(\Omega)$. We define that a function $u \in H^1(\Omega)$ is a weak solution of the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}, \tag{1}$$

if

$$\int_\Omega Du \cdot Dv \, dx = \int_\Omega fv \, dx$$

for all $v \in H^1(\Omega)$.

(a) (4%) Show that if (1) has a weak solution, then $\displaystyle\int_\Omega f \, dx = 0$.

(b) (4%) Define $H = \{w = u - (u)_\Omega : \ u \in H^1(\Omega)\}$. Show that $\|w\|_H := \|u - (u)_\Omega\|_{H^1}$ is a norm on H.

(c) (12%) Show that if $\displaystyle\int_\Omega f = 0$, then (1) has a weak solution by using Lax-Milgram Theorem to show that for all $z \in H$,

$$\int_\Omega Dw \cdot Dz \, dx = \int_\Omega fz \, dx.$$

4. (a) Show that the function $f(x) = \dfrac{1}{|x|^\alpha}$ lies in $W^{1,p}(\Omega)$ if and only if $\alpha < \frac{n-p}{p}$.

(b) Show that there is <u>no</u> constant $C > 0$ such that for all $u \in H^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |u|^2 \leq C \int_{\mathbb{R}^n} |Du|^2.$$

5. Find all the critical points (or equilibrium points) of the following system and determine the stability of each critical point:

$$\begin{aligned} \frac{dx}{dt} &= x(1 - 0.5y), \\ \frac{dy}{dt} &= y(-0.75 + 0.25x). \end{aligned}$$

End of Paper