Qualifying Examination-Differential Equations
National Sun Yat-sen University
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1. Solve the Cauchy problem

$$\begin{cases} xu_x + yuu_y = -xy \\ u = 5 \quad \textit{on} \quad xy = 1 \ (x > 0). \end{cases} \qquad (15 \ \textit{points})$$

2. Solve

$$\begin{cases} u_t(x,t) = \kappa u_{xx}(x,t) + F(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u(x,0) = f(x), \quad 0 < x < \ell \\ u(0,t) = u(\ell,t) = 0, \quad t > 0, \end{cases}$$

where f and F are given sufficient smooth functions. (15 points)

3. Let $u(x) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of

$$\triangle u + \sum_{k=1}^{n} a_k(x) u_{x_k} + c(x) u = 0,$$

where $c(x) < 0$ in Ω. Show that $u = 0$ on $\partial\Omega$ implies $u = 0$ in Ω. (15 points)

4. Suppose that

$$x' = f(t,x), \qquad x(\tau) = \xi,$$

where $f \in C(D \subset \mathbb{R} \times \mathbb{R}, \mathbb{R})$; D is an open connected set.

Give a sufficient condition for the existence of the local solution of the above differential equation. Prove your statement. (15 points)

5. (a) Show that the system of differential equations

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

has a solution which tends to ∞ as $t \to -\infty$ and tends to the origin as $t \to +\infty$. (10 points)

(b) Consider the system of differential equations

$$\begin{cases} \frac{dx}{dt} = y + x(1 - x^2 - y^2) \\ \frac{dy}{dt} = -x + y(1 - x^2 - y^2). \end{cases}$$

Show that for any x_0 and y_0, there is a unique solution $(x(t), y(t))$ defined for all $t \in \mathbb{R}$ such that $x(0) = x_0, \quad y(0) = y_0$. (10 points)

6. Discuss the stability of the zero solution of the following two systems.

(a)

$$\begin{cases} x' = e^{-x-2y} - 1 \\ y' = -x(1-y)^2 \end{cases} \qquad (10 \ \ points)$$

(b)

$$\begin{cases} x' = \lambda x - xy^2 \\ y' = 2x^2 y \end{cases} \quad , \quad (10 \ \ points)$$

where $\lambda \in \mathbb{R}$ is a parameter.

7. (a) Find all eigenvalues and eigenfunctions for the problem

$$\begin{cases} -u''(x) = \lambda u(x), \quad 0 < x < \ell, \\ u'(0) = u'(\ell) = 0. \end{cases} \qquad (10 \ \ points)$$

(b) Show that every solution of the Hermite differential equation

$$y'' - 2xy' + 2ay = 0 (a \geq 0)$$

has at most finitely many zeros in $\mathbb{R}$. (10 points)

End of the paper