Qualifying Examination-Differential Equations
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Each problem carries 20%.

1. Let D be an open set in $R \times R^{n+1}$ with an element of D written as (t, x), and $f : D \to R^n$ be a continuous function. Consider the following differential equation

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases} \tag{1}$$

   (a) For any $(t_0, x_0) \in D$ there is at least one solution of (1) passing through (t_0, x_0). Outline the steps of its proof.

   (b) If, in addition, $f(t, x)$ is locally lipschotzian with respect to x in D, then prove that for any (t_0, x_0) in D, there exists a unique solution $x(t, t_0, x_0)$ of (1) passing through (t_0, x_0).

2. (a) Find the general solution of

$$\frac{dX(t)}{dt} = AX(t), \text{ where } A = \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}. \tag{2}$$

   (b) Does every solution $Y(t)$ of equation (2) satisfy : $\lim_{t\to\infty} Y(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$? Why ?

3. Consider the following Predator-Prey system

$$\begin{cases} \dfrac{dx}{dt} = x(\gamma(1 - \dfrac{x}{K}) - \dfrac{my}{a+x}) \\ \dfrac{dy}{dt} = (\dfrac{mx}{a+x} - d)y, \\ x(0) > 0, y(0) > 0, \end{cases} \tag{3}$$

   where γ, K, m, a and d are positive constants. Find all equilibrium points with nonnegative components and do each stability analysis respectively

4. (a) Let Ω be a bounded, open, and connected set in R^n. If $u \in C^2(\Omega) \bigcap C^0(\overline{\Omega})$ and $\Delta u \geq 0$ in Ω, then state the Maximum Principle.

   (b) Let u be a solution of

$$\Delta u = u^3 - u \tag{4}$$

   on a bounded domain Ω. Assume that $u = 0$ on $\partial\Omega$. Show that $u \in [-1, 1]$ throughout Ω. Can the value ± 1 be achieved ?

5. Let $f \in C^2(R)$ and $g \in C^1(R)$. Prove that the solution of the initial value problem

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 \text{ for } x \in R, \ t > 0, \\ u(x,0) = f(x), u_t(x,0) = g(x), \ x \in R, \end{cases} \tag{5}$$

   is given by $u(x,t) = \frac{1}{2}\{f(x+t) + f(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} g(y)\, dy$.