Ph.D. Qualifying Examination: Algebra (Feb. 2010)

Notes. Let $\mathbf{C}$ denote the field of complex numbers, $\mathbf{R}$ the field of real numbers, $\mathbf{Q}$ the field of national numbers, $\mathbf{Z}$ the ring of integers, $\mathrm{Aut}(H)$ the automorphism group of the group H and $Z(G)$ the center of the group G.

[1] Let G be a finite group and let p be the smallest prime dividing the order of G. Prove:

(a) $\mathrm{Aut}(H) \cong \mathbf{Z}/p\mathbf{Z} \setminus \{0\}$ if H is a subgroup of G of order p. (10 points)

(b) If H is a normal subgroup of order p, then $H \subseteq Z(G)$. (10 points)

[2] Let $i = \sqrt{-1}$ in $\mathbf{C}$, the field of complex numbers, $\mathbf{R}$ the field of real numbers, and let x be an indeterminate.

(a) Show that the three additive groups $\mathbf{R} \oplus \mathbf{R}$, $\mathbf{R}[i]$, and $\mathbf{R}[x]/(x^2)$ are all isomorphic to each other. (10 points)

(b) Show that no two of the three rings $\mathbf{R} \oplus \mathbf{R}$, $\mathbf{R}[i]$, and $\mathbf{R}[x]/(x^2)$ are isomorphic to each other. (10 points)

[3] Let V be a vector space over $\mathbf{R}$ with $\dim_{\mathbf{R}} V \geq 2$. Suppose that $f\colon V \to V$ is a nonzero linear transformation satisfying $f(v) \in \mathbf{R}\,v$ for all $v \in V$. Prove that there exists $\beta \in \mathbf{R}$ such that $f(v) = \beta v$ for all $v \in V$. (10 points)

[4] Let $a, b \in \mathrm{M}_7(\mathbf{R})$ be such that $ab^{19} = 0$. Prove that $ab^7 = 0$. (10 points)

[5] Let $f(x) \in \mathbf{Q}[x]$ be of degree $n > 1$. Suppose that $m > 1$ is a square-free positive integer and $a, b \in \mathbf{Q}$. Prove that if $f(a + b\sqrt{m}) = 0$ then $f(a - b\sqrt{m}) = 0$. (10 points)

[6] Let $A \in \mathrm{End}_F(V)$, where V is a finite-dimensional vector space over the field F. If $q(x) \in F[x]$ is irreducible and if $q(A)$ is not one-to-one, prove that $q(x)$ divides the minimal polynomial of A. (10 points)

[7] (a) Prove that $\left[\mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})\colon \mathbf{Q}\right] = 8$. (10 points)

(b) Find the Galois group of $\mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbf{Q}$. (10 points)