Qualifying Examination in Algebra(Sep. 2008)

1. (20 points) Let n be a positive integer. Prove or disprove:

   (a) if n is a prime, then $n \mid a^{n-1} - 1$ for all integer a with $n \nmid a$.

   (b) if $n \mid a^{n-1} - 1$ for all integer a with $n \nmid a$, then n is a prime.

2. (10 points) Assume G is a group of order $p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t}$ where $p_1 < p_2 < \cdots < p_t$ are prime. Let H be a subgroup of G with index p_1. Prove or disprove that H is a normal subgroup of G.

3. (30 points) Let p be a prime. Prove or disprove:

   (a) if G is a group of order p, then G is abelian.

   (b) if G is a group of order p^2, then G is abelian.

   (c) if G is a group of order p^3, then G is abelian.

4. (20 points)

   (a) Let R be a commutative ring with identity. Show that if a is invertible in R and b is nilpotent(i.e. $b^n = 0$ for some positive integer n), then $a+b$ is invertible in R.

   (b) Is "commutative" essential in the statement (a)?

5. (10 points) Let n be a positive integer. Prove or disprove:

   $\mathbb{Z}[\sqrt{-n}]$ is a unique factorization domain **only if** $n = 1$ or $n = 2$.

6. (10 points) Let F be a field and K be an extension field of F. Assume a and b in K are algebraic over F. Prove or disprove that $a + b$ is algebraic over F.