PhD Qualifying Exam: Algebra, September 2007

Work out all of the problems and show details of your works.

[10%] 1. Let G be a finite group acting on a set S. Let $s \in S$ be fixed and denote $G_s = \{g \in G \mid g \cdot s = s\}$, the isotropy group of s. Show that the order of the orbit $Gs = \{g \cdot s \mid g \in G\}$ is equal to the index $[G : G_s]$ of the subgroup G_s in G.

[15%] 2. Let G be a group of order p^3 where p is a prime number. Suppose that G is not abelian.

(a) Show that the center of G has order p.

(b) Let Z be the center of G. Show that G/Z is isomorphic to the direct sum $\mathbb{Z}_p \times \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the cyclic group of order p.

[10%] 3. Let R be a ring and I an ideal in R. Let $[R : I] = \{r \in R \mid xr \in I \text{ for all } x \in R\}$. Show that $[R : I]$ is a two-sided ideal of R containing I.

[15%] 4. (a) Let R a ring and $a \in R$. Let I be the ideal generated by a. What is a typical element in I?

(b) Show that in $\mathbb{Q}[x]$, every ideal is generated by a single element.

[10%] 5. Let D be a principal ideal domain. Show that I is nonzero prime ideal in D if and only if it is a maximal ideal.

[15%] 6. Let F be a finite field. Show that the order of F is a power of a prime.

[10%] 7. Let F be an extension field of the field K. Show that

(a) If $[F : K]$ is prime, then there are no intermediate fields between F and K.

(b) If $u \in F$ has degree n over K, then n divides $[F : K]$.

[15%] 8. Let $R = \mathbb{Z}[i]$, a subring of the complex number field $\mathbb{C}$. Let M be an abelian group. Prove that M is an R-module if and only if there is exists a homomorphism $\varphi : M \to M$ such that $\varphi^2(x) = -x$ for all $x \in M$.