Qualifying Examination in Algebra

Sep. 2006

1. Prove or disprove:

   (a) if p is a prime, then p is a divisor of $(p-1)!+1$.

   (b) if p is a divisor of $(p-1)!+1$, then p is a prime.

2. Classify all the groups which has order 10. And prove it.

3. Prove or disprove: if R is a finite integral domain, then R is a field.

4. Let R be a commutative ring and $f(x) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$. Prove or disprove: if $f(x)$ is nilpotent, then a_i is nilpotent for all $i = 0, 1, \cdots, n$.

5. Let $\alpha_1, \alpha_2, \cdots, \alpha_t$ be roots of $f(X) \in \mathbb{Q}[X]$. Prove or disprove:

$$\mathbb{Q}(\alpha_1, \alpha_2, \cdots, \alpha_t) = \mathbb{Q}[\alpha_1, \alpha_2, \cdots, \alpha_t]$$

6. Find a polynomial $f(X) \in \mathbb{Q}[X]$ with degree 3 such that the Galois group of $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$ is isomorphism to $\mathbb{Z}_3$(where $\alpha_1, \alpha_2, \alpha_3$ are the three roots of $f(X)$). And prove it(i.e. prove the polynomial you give satisfy the desired properties).

1, 2: 20 points; 3–6: 15 points.