時間：中午 12:10~14:10

共十一題，配分：

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.
8%	8%	9%	7%	10%	8%	9%	16%	12%	5%	8%

答題時，每題都必須寫下題號與詳細步驟。

請依題號順序作答，不會作答題目請寫下題號並留空白。

1. (8%) Find the point $a > 0$ satisfying $\frac{1}{2}$

$$\int_1^a \frac{2}{t}\,\mathrm{d}t = \int_a^{\frac{1}{8}} \frac{1}{t}\,\mathrm{d}t.$$

解答: Applying the definition of the function $\ln x$ to the equation shows that

$$2\ln a = \ln\frac{1}{8} - \ln a. \qquad [3\%]$$

Simplifying the above equation gives

$$3\ln a = \ln\frac{1}{8}.$$

Since $3\ln a = \ln a^3$ [3%], we arrive at $a^3 = \frac{1}{8}$, namely, $a = \frac{1}{2}$ [2%]. □

2. (8%) Find the equation of the tangent line to $y = \log_{10}(2x)$ at $(5, 1)$. $y = \frac{1}{5\ln 10}(x-5)+1$

解答: Writing the equation $y = \log_{10}(2x)$ as

$$y = \log_{10} 2 + \log_{10} x = \log_{10} 2 + \frac{\ln x}{\ln 10},$$

it follows that

$$y'(x) = \frac{1}{x\ln 10}. \qquad [3\%]$$

This implies that the slope of the tangent line is

$$y'(5) = \frac{1}{5\ln 10}. \qquad [3\%]$$

Therefore, the tangent line is

$$y = \frac{1}{5\ln 10}(x-5) + 1. \qquad [2\%]$$

□

3. (9%) Find the indefinite integral $\ln(e^x + e^{-x}) + C$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,\mathrm{d}x.$$

解答: 【解法一】 Let $u = e^x + e^{-x}$ [3%]. Since $\mathrm{d}u = (e^x - e^{-x})\,\mathrm{d}x$ [2%], it follows that

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{u} = \ln|u| + c = \ln(e^x + e^{-x}) + C. \quad [4\%]$$

【解法二】

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,\mathrm{d}x = \int \frac{1}{e^x + e^{-x}}\,\mathrm{d}(e^x + e^{-x}) = \ln(e^x + e^{-x}) + C. \quad [9\%]$$ □

4. (7%) Evaluate $\lim_{x\to\infty}\left[\frac{1}{x}\cdot\frac{a^x - 1}{a - 1}\right]^{1/x}$, where $a > 0$, $a \neq 1$. a, if $a > 1$; 1, if $0 < a < 1$

解答: Let $y = \left[\frac{1}{x}\cdot\frac{a^x - 1}{a - 1}\right]^{1/x}$. For $a > 1$ and $x > 0$,

$$\ln y = \frac{1}{x}\left[\ln\frac{1}{x} + \ln(a^x - 1) - \ln(a - 1)\right] = -\frac{\ln x}{x} + \frac{\ln(a^x - 1)}{x} - \frac{\ln(a - 1)}{x}. \quad (1\%)$$

As $x \to \infty$. $\frac{\ln x}{x} \to 0$ (1%), $\frac{\ln(a-1)}{x} \to 0$ (1%), and

$$\frac{\ln(a^x - 1)}{x} = \frac{\ln\left[(1 - a^{-x})a^x\right]}{x} = \frac{\ln(1 - a^{-x})}{x} + \ln a \to \ln a. \quad (2\%)$$

So, $\ln y \to \ln a$. For $0 < a < 1$ and $x > 0$,

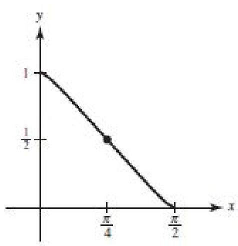
$$\ln y = \frac{-\ln x}{x} + \frac{\ln(1 - a^x)}{x} - \frac{\ln(1 - a)}{x} \to 0 \text{ as } x \to \infty. \quad (2\%)$$

Combining these results,

$$\lim_{x\to\infty} y = \begin{cases} a, & a > 1 \\ 1, & 0 < a < 1. \end{cases}$$ □

5. (10%) Evaluate the integral $\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{\sqrt{2}}}\,\mathrm{d}x$. $\frac{\pi}{4}$

解答:



$y = \frac{1}{1 + (\tan x)^{\sqrt{2}}}$ symmetric with respect to point $\left(\frac{\pi}{4}, \frac{1}{2}\right)$. (3%)

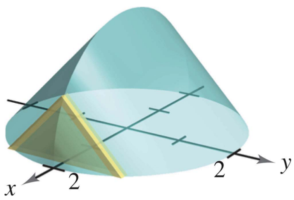
Let $I = \int_0^{\pi/2} \frac{1}{1+(\tan x)^{\sqrt{2}}}\,\mathrm{d}x$. Substitute $x = \pi/2 - y$ into I. (1%) Then

$$\begin{aligned} I &= \int_{\pi/2}^{0} \frac{1}{1+(\tan(\pi/2-y))^{\sqrt{2}}}(-\,\mathrm{d}y) && (1\%) \\ &= \int_0^{\pi/2} \frac{1}{1+(\cot y)^{\sqrt{2}}}\,\mathrm{d}y && (1\%) \\ &= \int_0^{\pi/2} \frac{(\tan y)^{\sqrt{2}}}{1+(\tan y)^{\sqrt{2}}}\,\mathrm{d}y && (1\%) \\ &= \int_0^{\pi/2} \frac{(\tan x)^{\sqrt{2}}}{1+(\tan x)^{\sqrt{2}}}\,\mathrm{d}x && (1\%) \end{aligned}$$

Add both I, then

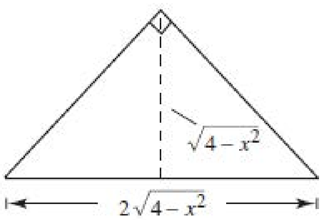
$$\begin{aligned} 2I &= \int_0^{\pi/2} \frac{1+(\tan x)^{\sqrt{2}}}{1+(\tan x)^{\sqrt{2}}}\,\mathrm{d}x \\ &= \frac{\pi}{2} && (1\%) \\ I &= \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4} && (1\% \ \square) \end{aligned}$$

6. (8%) Find the volumes of the solid whose base is bounded by the circle $x^2 + y^2 = 4$, with the indicated cross section taken perpendicular to the x-axis. Isosceles right triangles $\frac{32}{3}$



解答: $A(x) = \frac{1}{2}bh = 4 - x^2$ (2%)

$$\begin{aligned} V &= \int_{-2}^{2} (4-x^2)\,\mathrm{d}x && (2\%) \\ &= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} && (2\%) \\ &= \frac{32}{3} && (2\% \ \square) \end{aligned}$$



7. (9%) Use the shell method to write and evaluate the definite integral that represents the volume of the solid generated by revolving the plane region constitute by the $y=0$, $y=x$ and $x=2$ about the x-axis. $\dfrac{8\pi}{3}$

解答: $p(y)=y$ (2%), $h(y)=2-y$ (2%),

$$\begin{aligned} V &= 2\pi\int_0^2 y(2-y)\,\mathrm{d}y \\ &= 2\pi\int_0^2 \left(2y-y^2\right)\,\mathrm{d}y \\ &= 2\pi\left[y^2-y^3/3\right]_0^2 && (3\%) \\ &= \frac{8\pi}{3} && (2\%) \end{aligned}$$

8. (a) (8%) Find the indefinite integral $\displaystyle\int\left[z^2+\frac{1}{(1-z)^6}\right]\mathrm{d}z$. $\dfrac{z^3}{3}+\dfrac{(1-z)^{-5}}{5}+C$

(b) (8%) Find the indefinite integral $\displaystyle\int(\ln x)^2\,\mathrm{d}x$. $x\left((\ln x)^2-2\ln x+2\right)+C$

解答:

(a)

$$\begin{aligned} \int\left[z^2+\frac{1}{(1-z)^6}\right]\mathrm{d}z &= \int\left[z^2+(1-z)^{-6}\right]\,\mathrm{d}z && (4\%) \\ &= \frac{z^3}{3}+\frac{(1-z)^{-5}}{5}+C && (4\%) \end{aligned}$$

(b) Let $u=(\ln x)^2$, $\mathrm{d}u=2\ln x\cdot\dfrac{1}{x}$, $\mathrm{d}v=\mathrm{d}x$, $v=x$ (4%).

$$\begin{aligned} \int(\ln x)^2\,\mathrm{d}x &= (\ln x)^2\cdot x-\int x\cdot(2\ln x)\cdot\frac{1}{x}\,\mathrm{d}x \\ &= (\ln x)^2\cdot x-2\int\ln x\,\mathrm{d}x \\ &= (\ln x)^2\cdot x-2(x\ln x-x)+C \\ &= x\left((\ln x)^2-2\ln x+2\right)+C && (4\%\ \Box) \end{aligned}$$

9. (a) (6%) Evaluate the integral $\displaystyle\int_0^{2\sqrt{3}}\frac{x^2}{\sqrt{16-x^2}}\,\mathrm{d}x$. $\dfrac{8\pi}{3}-2\sqrt{3}$

(b) (6%) Evaluate the integral $\displaystyle\int\frac{x-9}{x^2+3x-10}\,\mathrm{d}x$. $2\ln|x+5|-\ln|x-2|+C$

解答:

(a) Let $x=4\sin\theta\Rightarrow\mathrm{d}x=4\cos\theta\,\mathrm{d}\theta$ $\cdots$(1%)

$$\int_0^{2\sqrt{3}}\frac{x^2}{\sqrt{16-x^2}}\,\mathrm{d}x=\int_0^{\pi/3}16\sin^2\theta\,\mathrm{d}\theta \qquad (1\%)$$

$$= \int_0^{\pi/3} 8(1-\cos 2\theta)\,d\theta \qquad (1\%)$$
$$= (8\theta - 4\sin 2\theta)\Big|_0^{\pi/3} \qquad (1\%)$$
$$= \frac{8\pi}{3} - 4\sin\left(\frac{2\pi}{3}\right) \qquad (1\%)$$
$$= \frac{8\pi}{3} - 2\sqrt{3} \qquad (1\%)$$

(b) $\dfrac{x-9}{x^2+3x-10} = \dfrac{x-9}{(x+5)(x-2)} = \dfrac{A}{x+5} + \dfrac{B}{x-2} \quad \cdots (1\%)$

Find $A = 2$ and $B = -1 \quad \cdots (1\% + 1\%)$

$$\int \frac{x-9}{x^2+3x-10}\,dx = \int \left(\frac{2}{x+5} + \frac{-1}{x-2}\right) dx$$
$$= 2\ln|x+5| - \ln|x-2| + C \qquad (1\%+1\%+1\%)$$

10. (5%) Find the value p for which the integral $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges and evaluate the integral for the value p. If $p<1$, the limit converges; $\dfrac{1}{1-p}$

解答: (5%) Since $\dfrac{1}{x^p}$ is continuous on $(0,1]$, $\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^p}\,dx. \quad \cdots (1\%)$

$$\int_t^1 \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{1-p}[1-t^{1-p}] & \text{if } p \neq 1 \quad \cdots (1\%) \\ -\ln t & \text{if } p = 1 \quad \cdots (1\%) \end{cases}$$

When $p<1$, the limit converges and $\cdots (1\%)$

$\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \frac{1}{1-p}[1-t^{1-p}] = \frac{1}{1-p} \quad \cdots (1\%).$ □

11. Let $f(x) = \dfrac{x^2}{x^6+9}$.

(a) (5%) Evaluate the integral $\displaystyle\int f(x)\,dx$. $\dfrac{1}{9}\arctan\left(\dfrac{x^3}{3}\right) + C$

(b) (3%) Evaluate the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$. $\dfrac{\pi}{9}$

解答:

(a) (5%) Let $x^3 = 3\tan\theta \Rightarrow 3x^2\,dx = 3\sec^2\theta\,d\theta. \quad \cdots (1\%)$

$$\int \frac{x^2}{x^6+9}\,dx = \int \frac{\sec^2\theta\,d\theta}{9\tan^2\theta + 9} \qquad (1\%)$$

$$= \frac{1}{9}\int d\theta \qquad (1\%)$$

$$= \frac{1}{9}\theta + C \qquad (1\%)$$

$$= \frac{1}{9}\arctan\left(\frac{x^3}{3}\right) + C \qquad (1\%)$$

(b) (3%)

$$\int_0^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_0^t f(x)\,\mathrm{d}x = \lim_{t\to\infty}\frac{1}{9}\Big[\arctan(\frac{t^3}{3}) - \arctan 0\Big] = \frac{\pi}{18} \qquad (1\%)$$

$$\int_{-\infty}^0 f(x)\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^0 f(x)\,\mathrm{d}x = \lim_{t\to-\infty}\frac{1}{9}\Big[\arctan 0 - \arctan\left(\frac{t^3}{3}\right)\Big] = \frac{\pi}{18} \qquad (1\%)$$

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \frac{\pi}{18} + \frac{\pi}{18} = \frac{\pi}{9} \qquad (1\% \ \square)$$

~全卷完~