時間：下午 7:00∼9:00

共十題，配分：

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
9%	8%	8%	5%	10%	10%	9%	8%	8%	25%

答題時，每題都必須寫下題號與詳細步驟。

請依題號順序作答，不會作答題目請寫下題號並留空白。

1. (9%) Determine whether the function

$$f(x) = \int_2^x \frac{\mathrm{d}t}{\sqrt{1+t^4}}$$

has an inverse on $\mathbb{R}$. If it does, find $(f^{-1})'(0)$. Yes; $\sqrt{17}$

解答: Since f is differentiable and $f'(x) = 1/\sqrt{1+x^4} > 0$ on $\mathbb{R}$, f is strictly increasing on $\mathbb{R}$ [3%]. Thus, f has an inverse on $\mathbb{R}$ [1%]. Using the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \qquad [3\%]$$

and the value $f^{-1}(0) = 2$ [1%], we have

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \sqrt{17}. \qquad (1\% \ \square)$$

2. (8%) Find the integral $\frac{1}{3\ln 3}\ln(1+3^{3x}) + C$

$$\int \frac{3^{3x}}{1+3^{3x}}\,\mathrm{d}x.$$

解答: Using the derivative $\frac{d}{\mathrm{d}x}3^{3x} = (3\ln 3)3^{3x}$ [4%] yields

$$\int \frac{3^{3x}}{1+3^{3x}}\,\mathrm{d}x = \frac{1}{3\ln 3}\ln(1+3^{3x}) + C. \qquad (4\%))$$

Alternative answer: Let $u = 1+3^{3x}$ [2%]. Then $du = (3\ln 3)3^{3x}\,\mathrm{d}x$ [2%]. Hence the method of substitution gives

$$\int \frac{3^{3x}}{1+3^{3x}}\,\mathrm{d}x = \frac{1}{3\ln 3}\int \frac{du}{u} = \frac{1}{3\ln 3}\ln u = \frac{1}{3\ln 3}\ln(1+3^{3x}) + C. \qquad (4\% \ \square)$$

3. (8%) Find the limit $\sqrt{e}$

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{n+\frac{1}{n}}.$$

解答: Using the fact

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}, \qquad [2\%]$$

one has

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{n}=\lim_{n\to\infty}\left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{1/2}=\left(\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}\right)^{1/2}=\sqrt{e}.\quad [2\%]$$

On the other hand, the limits $\lim_{n\to\infty}(1+1/(2n))=1$ and $\lim_{n\to\infty}1/n=0$ yield

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{\frac{1}{n}}=1^0=1.\quad [2\%]$$

Consequently, we arrive at

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{n+\frac{1}{n}}=\left[\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{n}\right]\cdot\left[\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{\frac{1}{n}}\right]=\sqrt{e}\cdot 1=\sqrt{e}.\quad [2\%]$$

Alternative answer: Note first that we have

$$\left(1+\frac{1}{2x}\right)^{x+\frac{1}{x}}=\exp\left[\left(x+\frac{1}{x}\right)\ln\left(1+\frac{1}{2x}\right)\right].\quad [2\%]$$

Then the L'Hôpital rule shows that

$$\lim_{x\to\infty}x\ln\left(1+\frac{1}{2x}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{\frac{-1}{2}x^{-2}}{1+\frac{1}{2x}}}{-x^{-2}}=\frac{1}{2}.\quad [2\%]$$

On the other hand,

$$\lim_{x\to\infty}\frac{1}{x}\ln\left(1+\frac{1}{2x}\right)=0.\quad [2\%]$$

Consequently, we have

$$\lim_{x\to\infty}\left(x+\frac{1}{x}\right)\ln\left(1+\frac{1}{2x}\right)=\frac{1}{2},\quad [1\%]$$

which gives the desired limit $e^{1/2+0}=e^{1/2}$ [1%]. □

4. (5%) Find the limit $\lim\limits_{x\to 0}\frac{\ln(\cos(ax))}{\ln(\cos(bx))}$, where a, b are constant. $\frac{a^2}{b^2}$

解答: 【解法一】

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(\cos(ax))}{\ln(\cos(bx))}&=\lim_{x\to 0}\frac{\frac{1}{\cos(ax)}(\cos(ax))'}{\frac{1}{\cos(bx)}(\cos(bx))'}&&\text{(利用羅畢達規則)}\quad(2\%)\\&=\lim_{x\to 0}\frac{\cos(bx)\cdot(-a\sin(ax))}{\cos(ax)\cdot(-b\sin(bx))}&&\text{(利用羅畢達規則)}\quad(2\%)\\&=\lim_{x\to 0}\frac{-b\sin(bx)\cdot(-a\sin(ax))+\cos(bx)\cdot(-a^2\cos(ax))}{-a\sin(ax)\cdot(-b\sin(bx))+\cos(ax)\cdot(-b^2\cos(bx))}\end{aligned}$$

$$= \frac{0 + 1 \cdot (-a^2)}{0 + 1 \cdot (-b^2)}$$
$$= \frac{a^2}{b^2} \qquad (1\%)$$

【解法二】

$$\lim_{x\to 0} \frac{\ln(\cos(ax))}{\ln(\cos(bx))} = \lim_{x\to 0} \frac{\frac{-\sin(ax)}{\cos(ax)} \cdot a}{\frac{-\sin(bx)}{\cos(bx)} \cdot b} \qquad \text{(利用羅畢達規則)} \qquad (2\%)$$
$$= \lim_{x\to 0} \frac{a\tan(ax)}{b\tan(bx)} \qquad \text{(利用羅畢達規則)} \qquad (2\%)$$
$$= \lim_{x\to 0} \frac{a^2\sec^2(ax)}{b^2\sec^2(bx)}$$
$$= \frac{a^2 \cdot 1}{b^2 \cdot 1}$$
$$= \frac{a^2}{b^2} \qquad (1\% \ \square)$$

5. (a) (5%) Use implicit differentiation to find an equation $\arcsin x + \arcsin y = \frac{\pi}{2}$ of the tangent line to the graph of the equation at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. $y = -x + \sqrt{2}$

(b) (1%) Sketch the region whose area is represented by

$$\int_0^1 \arcsin x \, dx.$$

(c) (4%) Find the exact area of (b) analytically. $\frac{\pi}{2} - 1$

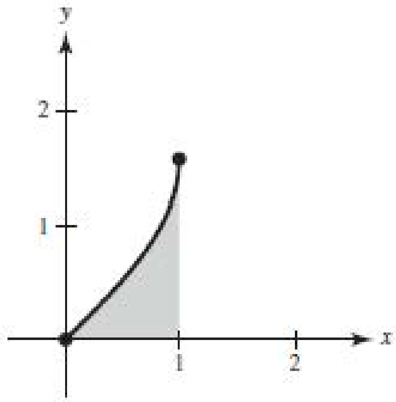
解答:

(a)

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} y' = 0 \qquad (1\%)$$
$$\frac{1}{\sqrt{1-y^2}} y' = \frac{-1}{\sqrt{1-x^2}} \qquad (1\%)$$

At $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$: $y' = -1$. (1%) Tangent line: $y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)$, $y = -x + \sqrt{2}$. (2%)

(b)



Shaded area is given by $\int_0^1 \arcsin x \, \mathrm{d}x$. (1%0

(c) 【解法一】

$$\int_0^1 \arcsin x \, \mathrm{d}x = x \arcsin x \Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x \qquad (2\%)$$
$$= \arcsin 1 - 1 \qquad (1\%)$$
$$= \frac{\pi}{2} - 1 \qquad (1\%)$$

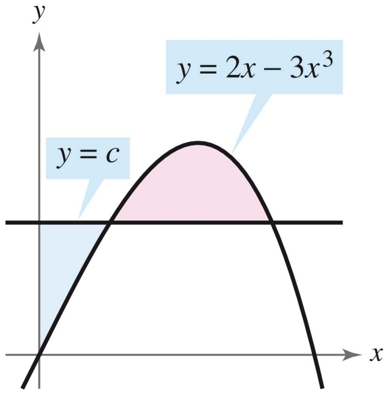
【解法二】 Divide the rectangle into two regions.

$$\text{Area rectangle} = (\text{base})(\text{height}) = 1\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \qquad (1\%)$$
$$\text{Area rectangle} = \int_0^1 \arcsin x \, \mathrm{d}x + \int_0^{\pi/2} \sin y \, \mathrm{d}y = \frac{\pi}{2} \qquad (2\%)$$
$$= \int_0^1 \arcsin x \, \mathrm{d}x + [-\cos y]_0^{\pi/2}$$
$$= \int_0^1 \arcsin x \, \mathrm{d}x + 1 \qquad (1\%)$$

So,

$$\int_0^1 \arcsin x \, \mathrm{d}x = \frac{\pi}{2} - 1, \quad (\approx 0.5708). \qquad (1\% \ \square)$$

6. (a) (5%) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as shown in the figure. Find c so that the areas of the two shaded regions are equal. $\frac{4}{9}$



(b) (5%) Find the volume of solid generated by revolving the curve $y = e^{-x}\sin x,\ x \geq$ 0, about the x-axis. $\frac{\pi}{8}$

解答:

(a) You want to find c such that:

$$\int_0^b \left[(2x - 3x^3) - c\right] \mathrm{d}x = 0 \qquad (1\%)$$

$$\left[x^2 - \frac{3}{4}x^4 - cx\right]_0^b = 0 \qquad (1\%)$$

$$b^2 - \frac{3}{4}b^4 - cb = 0$$

But, $c = 2b - 3b^3$ because (b, c) is on the graph

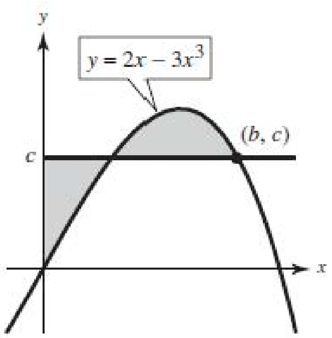
$$b^2 - \frac{3}{4}b^4 - (2b - 3b^3)b = 0 \qquad (1\%)$$

$$4 - 3b^2 - 8 + 12b^2 = 0$$

$$9b^2 = 4$$

$$b = \frac{2}{3} \qquad (1\%)$$

$$c = \frac{4}{9}. \qquad (1\%)$$



(b)

$$V = \int_0^\infty \pi \left(e^{-x}\sin x\right)^2 \mathrm{d}x \qquad (1\%)$$

$$
\begin{aligned}
&= \pi \int_0^{\infty} e^{-2x} (\sin x)^2 \, \mathrm{d}x \\
&= \pi \left[\left(-\frac{1}{2} e^{-2x} (\sin x)^2 \right) \Bigg|_0^{\infty} + \frac{1}{2} \int_0^{\infty} e^{-2x} \sin 2x \, \mathrm{d}x \right] \qquad (1\%) \\
&= \frac{\pi}{2} \int_0^{\infty} e^{-2x} \sin 2x \, \mathrm{d}x
\end{aligned}
$$

因此計算

$$
\begin{aligned}
\frac{\pi}{2} \int_0^{\infty} e^{-2x} \sin 2x \, \mathrm{d}x &= \frac{\pi}{2} \left[\left(-\frac{1}{2} e^{-2x} \cos 2x \right) \Bigg|_0^{\infty} - \int_0^{\infty} e^{-2x} \cos 2x \, \mathrm{d}x \right] \\
&= \frac{\pi}{2} \left[\left(\frac{1}{2} - \frac{1}{2} e^{-2x} \sin 2x \right) \Bigg|_0^{\infty} - \int_0^{\infty} e^{-2x} \sin 2x \, \mathrm{d}x \right] \\
&= \frac{\pi}{4} - \frac{\pi}{2} \int_0^{\infty} e^{-2x} \sin 2x \, \mathrm{d}x \qquad (1\%) \\
\Rightarrow \pi \int_0^{\infty} e^{-2x} \sin 2x \, \mathrm{d}x &= \frac{\pi}{4} \qquad (1\%)
\end{aligned}
$$

所以

$$
V = \frac{\pi}{2} \int_0^{\infty} e^{-2x} \sin 2x \, \mathrm{d}x = \frac{\pi}{2} \times \frac{1}{4} = \frac{\pi}{8} \qquad 1\% \ \square
$$

7. (9%) Verifying a Formula

   (a) (2%) Given a circular sector with radius L and central angle θ, show that the area of the sector is given by $S = \frac{1}{2} L^2 \theta$.

   (b) (3%) By joining the straight-line edges of the sector in part (a), a right circular cone is formed and the lateral surface area of the cone is the same as the area of the sector. Show that the area is $S = \pi r L$, where r is the radius of the base of the cone.

   (c) (4%) Use the result of part (b) to verify that the formula for the lateral surface area of the frustum of a cone with slant height L and radii r_1 and r_2 is $S = \pi(r_1 + r_2)L$.

解答:

(a) Area of circle with radius $L : A = \pi L^2$ (1%). Area of sector with central angle θ (in radians): $S = \frac{\theta}{2\pi} A = \frac{\theta}{2\pi} (\pi L^2) = \frac{1}{2} L^2 \theta$ (1%).

(b) Let s be the arc length of the sctor, which is the circumference of the base of the cone. Here $s = L\theta = 2\pi r$ (1%). Therefore, $S = \frac{1}{2} L^2 \theta = \frac{1}{2} L^2 \left(\frac{s}{L} \right) = \frac{1}{2} L s = \frac{1}{2} L (2\pi r) = \pi r L$ (2%).

(c) The lateral surface area of the frustum is the difference of the large cone and the small one. $S = \pi r_2 (L + L_1) - \pi r_1 L_1 = \pi r_2 L + \pi (r_2 - r_1) L_1$ (1%). By similar triangles, $\frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow r_1 L = (r_2 - r_1) L_1$ (2%). So, $S = \pi r_2 L + \pi (r_2 - r_1) L_1 = \pi r_2 L + \pi r_1 L = \pi (r_1 + r_2) L$ (1%). $\square$

8. (8%) Verifying a Reduction Formula by using integration by parts

(a) (4%) $\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$.

(b) (4%) $\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$.

解答:

(a) $dv = \cos x\ dx \Rightarrow v = \sin x$. $u = \cos^{n-1} x \Rightarrow du = -(n-1)\cos^{n-2} x \sin x\ dx$. (2%)
$\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,(1-\cos^2 x)\ dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$.
Therefore, $n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx$, $\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$. (2%)

(b) Let $u = \sec^{n-2} x$, $du = (n-2)\sec x \tan x$, $dv = \sec^2 x\ dx$, $v = \tan x$. (2%)
$\int \sec^n x\,dx = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1) = \sec^{n-2} x \tan x - (n-2)\left[\int \sec^n x\,dx - \int \sec^{n-2} x\,dx\right]$
$(n-1)\int \sec^n x\,dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx$. $\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$. (2%) □

9. (8%) Prove the following versions of Wallis's Formulas

(a) (4%) If n is odd ($n \geq 3$), then $\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$.

(b) (4%) If n is even ($n \geq 2$), then $\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$.

解答:

(a) $\int_0^{\pi/2} \cos^n x\,dx = \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx = \frac{n-1}{n}\left(\left[\frac{\cos^{n-3} x \sin x}{n-2}\right]_0^{\pi/2} + \frac{n-3}{n-2}\int_0^{\pi/2} \cos^n\right.$
$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\left(\left[\frac{\cos^{n-5} x \sin x}{n-4}\right]_0^{\pi/2} + \frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx\right) = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx$
$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\int_0^{\pi/2} \cos x\,dx = \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots(\sin x)\right]_0^{\pi/2} = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots 1 = (1)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right) = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$ (2%)

(b) $\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\int_0^{\pi/2} dx$ (2%) $= \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots x\right]_0^{\pi/2} = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{\pi}{2}. = \left(\frac{\pi}{2}\cdot\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$ (2%)
□

10. (a) (7%) Evaluate the definite integral $\displaystyle\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$. $2\left(\sqrt{3}-\frac{\pi}{3}\right)$

(b) (7%) Evaluate the definite integral $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$. $\frac{9}{2}$

(c) (11%) Evaluate the improper integral $\displaystyle\int_1^\infty \frac{x-1}{x^4+x^3+x^2+x}\,dx$. $\frac{\pi}{4}-\ln(2)$

解答:

(a) (7%) Let $x = 2\sec\theta$, $0 \leq \theta \leq \pi$, $\theta \neq \frac{\pi}{2}$. Then $dx = 2\sec\theta\tan\theta\,d\theta$,
$x = 2 \Rightarrow \theta = 0$, and $x = 4 \Rightarrow \theta = \frac{\pi}{3}$.

$$\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx = \int_0^{\frac{\pi}{3}} \frac{2\tan\theta}{2\sec\theta}(2\sec\theta\tan\theta\,d\theta) \qquad (3\%)$$

$$= 2\int_0^{\frac{\pi}{3}} \tan^2\theta \,\mathrm{d}\theta = 2\int_0^{\frac{\pi}{3}} (\sec^2\theta - 1)\,\mathrm{d}\theta \qquad (1\%)$$

$$= 2(\tan\theta - \theta)\Big|_0^{\frac{\pi}{3}} \qquad (1\%)$$

$$= 2\Big[\Big(\tan\frac{\pi}{3} - \frac{\pi}{3}\Big) - (\tan 0 - 0)\Big] \qquad (1\%)$$

$$= 2\left(\sqrt{3} - \frac{\pi}{3}\right) \qquad (1\%)$$

(b) (7%) $\displaystyle\int \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = \frac{3}{2}(x-1)^{\frac{2}{3}} \quad \cdots (1\%)$

Because the integrand has an infinite discontinuous at $x = 1$, then

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x \qquad (1\%)$$

$$\int_0^1 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = \lim_{t\to 1^-}\frac{3}{2}\Big\{(t-1)^{\frac{2}{3}} - 1\Big\} = -\frac{3}{2} \qquad (2\%)$$

$$\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = \lim_{t\to 1^+}\int_t^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = \lim_{t\to 1^+}\frac{3}{2}\Big\{4 - (t-1)^{\frac{2}{3}}\Big\} = 6 \qquad (2\%)$$

Then $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = -\frac{3}{2} + 6 = \frac{9}{2} \quad \cdots (1\%)$.

(c) (11%)

$$\frac{x-1}{x^4+x^3+x^2+x} = \frac{x-1}{x(x+1)(x^2+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \qquad (1\%)$$

$$= \frac{-1}{x} + \frac{1}{x+1} + \frac{1}{x^2+1} \qquad (3\%)$$

$$\int_1^\infty \frac{x-1}{x^4+x^3+x^2+x}\,\mathrm{d}x$$

$$= \lim_{t\to\infty}\int_1^t \left(\frac{-1}{x} + \frac{1}{x+1} + \frac{1}{x^2+1}\right)\mathrm{d}x \qquad (1\%)$$

$$= \lim_{t\to\infty}\left\{\ln\left(\left|\frac{t+1}{t}\right|\right) + \arctan(t) - \ln(2) - \frac{\pi}{4}\right\} \qquad (3\%)$$

$$= 0 + \frac{\pi}{2} - \ln(2) - \frac{\pi}{4} = \frac{\pi}{4} - \ln(2) \qquad (3\%\ \square)$$

～全卷完～