National Sun Yat-sen University
Department of Applied Mathematics

Ph.D. Qualifying Exam

Mathematical Statistics

INSTRUCTIONS:

- You have **240 minutes** to complete this exam which consists of **9 questions**.
- **Please show ALL your steps to receive partial credit.**
- Please organize your work in a reasonably neat and coherent way.
- Please label the answer clearly.
- Notations: cdf, cumulative distribution function; i.i.d., independent and identically distributed; pdf, probability density function; pmf, probability mass function;

Question	1	2	3	4	5	6	7	8	9	Total
Points	10	10	10	10	12	12	12	12	12	100
Score										

Question 1. State the following theorems for i.i.d. random variables. Please define all the notations you used as clear as possible.

(a) Weak law of large numbers.

(b) Central limit theorem.

Question 2. Let N be a Poisson random variable with pmf

$$P(N=n)=\frac{\xi^n e^{-\xi}}{n!}, \quad n=0,1,\ldots,$$

where $\xi>0$. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. exponential random variables with mean $1/\lambda$, where $\lambda>0$.

(a) Find the moment generating function of $S=X_1+\cdots+X_N$.

(b) Find $\text{Var}(S)$.

Question 3. Let $X_1, \ldots, X_n$ be i.i.d. Beta$(\theta, 1)$ random samples, where $\theta > 0$.

(a) Find the method of moments estimator for θ, denoted by $\hat{\theta}$.

(b) Find the asymptotic variance of $\hat{\theta}$.

Question 4. Let X be a Binomial(n, p) random variable, and let the success probability p has a Beta(α, β) prior distribution with pdf

$$\pi(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}, \quad 0 < p < 1,$$

where $\alpha > 0$ and $\beta > 0$ are hyperparameters and

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

is the gamma function. The number of trials n is a known positive integer.

(a) Find the posterior distribution of p.

(b) Calculate the posterior mean and variance.

Question 5. Let X and Y be i.i.d. exponential random variables with mean $1/\lambda$, where $\lambda > 0$. Let $T = \min(X, Y)$ and $\delta = I(X < Y)$, where I is the indicator function.

(a) Find the joint distribution of (T, δ).

(b) Let (T_i, δ_i), $i = 1, \ldots, n$ be i.i.d. random samples from (T, δ). Find the maximum likelihood estimator for λ.

Question 6. Let $X_1, \ldots, X_n$ be i.i.d. $\text{Unif}(0, \theta)$ random variables with $\theta > 0$. Construct a nonrandomized test of size α for testing

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0,$$

where $\theta_0 > 0$ is a pre-specified value. You need to verify that your test has size α.

Question 7. Let X be a single observation from the pmf

$$P(X = x \mid \theta) = \left(\frac{\theta}{3}\right)^{|x|/2} \left(1 - \frac{2\theta}{3}\right)^{1-|x|/2}, \quad x \in \{-2, 0, 2\},$$

where $0 \leq \theta \leq 1$ is an unknown parameter.

(a) Find the maximum likelihood estimator for θ, denoted by $\hat{\theta}$.

(b) Define the estimator

$$T(X) = \begin{cases} 3 & \text{if } X = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $T(X)$ is an unbiased estimator for θ.

(c) Find a better estimator than $T(X)$ and prove that it is better.

Question 8. Let X_1 and X_2 be i.i.d. gamma random variables from pdf

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0,$$

where $\lambda > 0$ and $\alpha > 0$ are parameters.

(a) Find the joint distribution of $Y_1 = X_1/(X_1 + X_2)$ and $Y_2 = X_1 + X_2$.

(b) Find $E|X_1 - X_2|$.

Question 9. Let X_1 and X_2 be i.i.d. shifted exponential random variables with pdf

$$f(x \mid \mu, \lambda) = \lambda e^{-\lambda(x-\mu)}, \quad x > \mu,$$

where $\mu > 0$ and $\lambda > 0$ are unknown parameters, respectively. Let $X_{(1)} < X_{(2)}$ be the order statistics of X_1 and X_2.

(a) Show that $X_{(1)} - \mu$ and $X_{(2)} - X_{(1)}$ are independent.

(b) Show that

$$R = \frac{2(X_{(1)} - \mu)}{X_{(2)} - X_{(1)}}$$

is a pivot.

(c) Construct a 95% confidence interval for μ based on R.

This is the end of the exam.