Topological method for coloring in Kneser graphs

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One of the earliest and most spectacular applications of topological methods in combinatorics is Lovász's 1978 proof of a conjecture of Kneser (1955). In term of graphs, for integers $k \geq 1$ and $n \geq 2k-1$, consider the Kneser graph $\mathrm{KG}(n,k)$ whose vertex set $V(n,k) = \{x \subseteq [n] \colon |x| = k\}$ and edge set $E(n,k) = \{xy \colon x, y \in V(n,k), x \cap y = \emptyset\}$, where $[n] = \{1, 2, \ldots, n\}$. By using Borsuk's theorem, Lovász proved that the chromatic number $\chi(\mathrm{KG}(n,k)) = n - 2k + 2$. Simple proofs were offered by Bárány (1978) and Green (2002), still using Borsuk's theorem. Matous'vek (2004) gave a combinatorial proof using the idea for proving Tucker's lemma. Circular chromatic number $\chi_c(G)$ of a graph G is a refinement of the chromatic number $\chi(G)$, as $\chi(G) = \lceil \chi_c(G) \rceil$. Johnson, Holroyd and Stahl (1997) first studied circular chromatic numbers of Kneser graphs, and conjectured that the equality $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$ always holds. This conjecture was confirmed by Hajiabolhassan and Zhu (2003) for fixed k with n sufficiently large; and by Meunier (2005) and Simonyi and Tardos (2006) independently for even n. Chen (2011) completely proved the Johnson-Holroyd-Stahl conjecture by using Fan's lemma. Chang, Liu and Zhu (2013) gave a short proof.