1. Let $X_1, X_2, \ldots$ be a sequence of random variable that converges in probability to a constant $a$. Assume that $P(X_i > 0) = 1$ for all $i$.

(a) Verify that the sequences defined by $Y_i = \sqrt{X_i}$ and $Y_i' = a/X_i$ converge in probability.

(b) If $S^2_n \rightarrow \sigma^2$ in probability, then prove that $\sigma/S_n$ converges in probability to 1 where $S^2_n = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$.

解答:

(a) For any $\epsilon > 0$,

$$P(|\sqrt{X_n} - \sqrt{a}| > \epsilon) = P(|\sqrt{X_n} - \sqrt{a}| \sqrt{X_n} + \sqrt{a} > \epsilon \sqrt{X_n} + \sqrt{a})$$

$$= P(|X_n - a| > \epsilon \sqrt{X_n} + \sqrt{a})$$

$$\leq P(|X_n - a| > \epsilon \sqrt{a}) \rightarrow 0,$$

as $n \rightarrow \infty$, since $X_n \rightarrow a$ in probability.

Thus $\sqrt{X_n} \rightarrow \sqrt{a}$ in probability.

For any $\epsilon > 0$,

$$P\left(\left|\frac{a}{X_n} - 1\right| \leq \epsilon\right) = P\left(\frac{a}{1+\epsilon} \leq X_n \leq \frac{a}{1-\epsilon}\right)$$

$$= P\left(a - \frac{a\epsilon}{1+\epsilon} \leq X_n \leq a + \frac{a\epsilon}{1-\epsilon}\right)$$

$$\geq P\left(a - \frac{a\epsilon}{1+\epsilon} \leq X_n \leq a + \frac{a\epsilon}{1+\epsilon}\right)$$

$$= P\left(|X_n - a| \leq \frac{a\epsilon}{1+\epsilon}\right) \rightarrow 1,$$ as $n \rightarrow \infty$, since $X_n \rightarrow a$ in probability.

Thus $a/X_n \rightarrow 1$ in probability.

(b) $S^2_n \rightarrow \sigma^2$ in probability. By (a), $S_n = \sqrt{S^2_n} \rightarrow \sqrt{\sigma^2} = \sigma$ in probability, $\sigma/S_n \rightarrow 1$ in probability.

2. Let $X_1, \ldots, X_n$ be a sequence of random variables that converges in distribution to a random variable $X$. Let $Y_n$ be a sequence of random variables with the property that for any finite number $c$,

$$\lim_{n \rightarrow \infty} P(Y_n > c) = 1.$$

Show that for any finite number $c$,

$$\lim_{n \rightarrow \infty} P(X_n + Y_n > c) = 1.$$
解答: For all $\epsilon > 0$ there exist $N$ such that if $n > N$, then $P(X_n + Y_n > c) > 1 - \epsilon$. Choose $N_1$ such that $P(X_n > -m) > 1 - \epsilon/2$ and $N_2$ such that $P(Y_n > c + m) > 1 - \epsilon/2$. Then

$$P(X_n + Y_n > c) \geq P(X_n > -m, Y_n > c + m) \geq P(X_n > -m) + P(Y_n > c + m) - 1 = 1 - \epsilon.$$

3. Given that $N = n$, the conditional distribution of $Y$ is $\chi^2_n$. The unconditional distribution of $N$ is Poisson($\theta$).

(a) Calculate $E[Y]$ and $\text{Var}(Y)$ (unconditional moments).
(b) Show that, as $\theta \to \infty$, $(Y - E[Y])/\sqrt{\text{Var}(Y)} \to N(0, 1)$ in distribution.

解答:

(a) $E[Y] = E(E(Y|N)) = E(2N) = 2\theta$ and $\text{Var}(Y) = \text{Var}(E(Y|N)) + E(\text{Var}(Y|N)) = \text{Var}(2N) + E(4N) = 8\theta$.
(b) The moment generating function of $Y$:

$$M_Y(t) = E(E(e^{itY}|N)) = E(1/(1 + 2it)^N) = e^{2\theta t/(1-2it)}.$$

Note that $M_{aY+b}(t) = e^{bt}M_Y(at)$. Thus

$$M_{(Y-E(Y))/\sqrt{\text{Var}(Y)}}(t) = e^{-\sqrt{\theta/2t}}e^{2\theta t/(1+2t/\sqrt{2\theta})} = e^{-\sqrt{\theta/2t}}e^{\sqrt{\theta/2}(1+t/\sqrt{2\theta}+o(t/\theta^2))} \to e^{-t^2/2} \text{ as } \theta \to \infty.$$

Then the desired property is proved.

4. Let $f(t)$ be a real-valued and continuous function which is defined for all real $t$ and which satisfies the following conditions:

(i) $f(0) = 1$
(ii) $f(-t) = f(t)$
(iii) $f(t)$ convex for $t > 0$
(iv) $\lim_{t \to \infty} f(t) = 0$.

Then $f(t)$ is the characteristic function of an absolutely continuous distribution function.

解答: Since $f(t)$ is a convex function it has everywhere a right-hand derivative which we denote by $f'(t)$. The function $f'(t)$ is non-decreasing for $t > 0$. It follow from (iv) that $f'(t) \leq 0$ for $t > 0$ and that

$$\lim_{t \to \infty} f'(t) = 0.$$

It is easily seen that the integral $\int_{-\infty}^{\infty} e^{-itx} f(t) dt$ exists for all $x \neq 0$. We write

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} f(t) dt. \quad (4.3.1)$$
We see from (ii) and (4.3.1) that
\[ p(x) = \frac{1}{\pi} \int_0^\infty f(t) \cos tx \, dt. \]

The condition of Fourier’s inversion theorem are satisfied and we obtain
\[ f(t) = \int_{-\infty}^\infty e^{itx} p(x) \, dx. \]

If follows from (i) that \( \int_{-\infty}^\infty p(x) \, dx = 1 \) and the proof of theorem 4.3.1 is completed as soon as we show that \( p(x) \) is non-negative.

Integrating by parts and writing \( g(t) = -f'(t) \) we get
\[ p(x) = \frac{1}{\pi x} \int_0^\infty g(t) \sin tx \, dt \tag{4.3.3} \]
where \( g(t) \) is a non-increasing, non-negative function for \( t > 0 \) while
\[ \lim_{t \to -\infty} g(t) = 0. \]

Then
\[ p(x) = \frac{1}{\pi x} \int_0^{\pi/x} \left[ \sum_{j=0}^\infty (-1)^j g \left( t + \frac{j\pi}{x} \right) \right] \sin tx \, dt. \]

Let \( x > 0 \); the series
\[ \sum_{j=0}^\infty (-1)^j g \left( t + \frac{j\pi}{x} \right) \]
is an alternating series whose terms are non-increasing is absolute value; since the first term of the series is non-negative one sees that the integrand id non-negative. Thus \( p(x) \geq 0 \) for \( x > 0 \). Formula (4.3.2) indicates that \( p(x) \) is an even function of \( x \) so that \( p(x) \geq 0 \) if \( x \neq 0 \). Therefore \( p(x) \) is a frequency function and \( f(t) \) is the characteristic function of the absolutely continuous distribution \( F(x) = \int_{-\infty}^x p(y) \, dy. \)

5. Let \( X \) be a nonnegative random variable. Prove that
\[ E[X] \leq (E[X^2])^{1/2} \leq (E[X^3])^{1/3} \leq \cdots \]

解答：\( g(x) = x^{n/(n-1)} \) is convex. Hence, by Jensen’s Inequality
\[ E[Y^{n/(n-1)}] \geq (E[Y])^{n/(n-1)}. \]

Now set \( Y = X^{n-1} \) and so
\[ E[X^n] \geq (E[X^{n-1}])^{n/(n-1)} \quad \text{or} \quad (E[X^n])^{1/n} \geq (E[X^{n-1}])^{1/(n-1)} \]

6. Evaluate
\[ \lim_{n \to \infty} e^{-n} \sum_{k=0}^n \frac{n^k}{k!} \]
解答：Let $X_i$ be Poisson with mean 1. Then

$$P\left\{ \sum_{i=1}^{n} X_i \leq n \right\} = P\left\{ \frac{\sum_{i=1}^{n} X_i - n}{\sqrt{n}} \leq 0 \right\} = e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$ 

But for $n$ large $\frac{\sum_{i=1}^{n} X_i - n}{\sqrt{n}}$ has approximately a standard normal distribution with mean 0 by central limit theorem, and so $\lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$.

7. Suppose that a population consists of a fixed number, say, $m$, of genes in any generation. Each gene is one of two possible genetic types. If any generation has exactly $i$ (of its $m$) genes being type 1, then the next generation will have $j$ type 1 (and $m-j$ type 2) genes with probability

$$\binom{m}{j} \left( \frac{i}{m} \right)^j \left( \frac{m-i}{m} \right)^{m-j}, \quad j = 0, 1, \ldots, m$$

Let $X_n$ denote the number of type 1 genes in the $n$th generation, and assume that $X_0 = i$.

(a) Find $E[X_n]$.

(b) What is the probability that eventually all the genes will be type 1?

解答:

(a) Given $X_n$, $X_{n+1}$ is binomial with parameters $m$ and $p = X_n/m$. Hence, $E[X_{n+1}|X_n] = m(X_n/m) = X_n$, and so $E[X_{n+1}] = E[X_n]$. So $E[X_n] = E[X_0] = i$ for all $n$.

(b) Note that as all states but 0 and $m$ are transient, it follows that $X_n$ will converge to either 0 or $m$. Hence, for $n$ large

$$E[X_n] = m \cdot P\{\text{hits } m\} + 0 \cdot P\{\text{hits } 0\} = m \cdot P\{\text{hits } m\}$$

But $E[X_n] = i$ and thus $P\{\text{hits } m\} = i/m$.

8. Consider a Markov chain with states 0, 1, 2, 3, 4. Suppose $p_{0,4} = 1$; and suppose that when the chain is in states $i$, $i > 0$ the next state is equally likely to be any of the states 0, 1, \ldots, $i - 1$. Find the limiting probabilities of this Markov chain.

解答: The equations are

$$\begin{align*}
\pi_0 &= \pi_1 + 1/2\pi_2 + 1/3\pi_3 + 1/4\pi_4 \\
\pi_1 &= 1/2\pi_2 + 1/3\pi_3 + 1/4\pi_4 \\
\pi_2 &= 1/3\pi_3 + 1/4\pi_4 \\
\pi_3 &= 1/4\pi_4 \\
\pi_4 &= \pi_0
\end{align*}$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

The solution is

$$\pi_0 = \pi_4 = 12/37, \quad \pi_1 = 6/37, \quad \pi_2 = 4/37, \quad \pi_3 = 3/37$$
9. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables. Show that the probability that the largest of them is greater than the sum of the others is $n/2^{n-1}$. That is, if

$$M = \max_j X_j$$

then show

$$P\left\{ M > \sum_{i=1}^{n} X_i - M \right\} = \frac{n}{2^{n-1}}$$

Hint: What is $P\{X_1 > \sum_{i=2}^{n} X_i\}$?

解答: To begin, note that

$$P\left\{ X_1 > \sum_{i=2}^{n} X_i \right\} = P\{X_1 > X_2\}P\{X_1 - X_2 > X_3\}P\{X_1 - X_2 - X_3 > X_4\} \cdots$$

$$\times P\{X_1 - X_2 - \cdots - X_{n-1} > X_n\} = (1/2)^{n-1}. \quad \text{(memoryless property)}$$

Hence,

$$P\left\{ M > \sum_{i=1}^{n} X_i - M \right\} = \sum_{i=1}^{n} P\left\{ X_1 > \sum_{j \neq i}^{n} X_i \right\} = n/2^{n-1}.$$

10. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda$, that is independent of the sequence $X_1, X_2, \ldots$ of independent and identically distributed random variables with mean $\mu$ and variance $\sigma^2$. Find

$$\text{Cov}\left( N(t), \sum_{i=1}^{N(t)} X_i \right)$$

解答:

$$E\left[ \sum_{i=1}^{N(t)} X_i \right] = E\left[ E\left[ \sum_{i=1}^{N(t)} X_i | N(t) \right] \right] = E[\mu N(t)] = \mu \lambda t$$

$$E\left[ N(t) \sum_{i=1}^{N(t)} X_i \right] = E\left[ E\left[ N(t) \sum_{i=1}^{N(t)} X_i | N(t) \right] \right] = E[\mu N^2(t)] = \mu (\lambda t + \lambda^2 t^2)$$

Therefore,

$$\text{Cov}\left( N(t), \sum_{i=1}^{N(t)} X_i \right) = \mu (\lambda t + \lambda^2 t^2) - \lambda t (\mu \lambda t) = \mu \lambda t.$$