條件機率與條件期望值

許湘伶(應數博93)

本單元介紹變數代換的概念, 其內容主要取材自黃文璋(2003a)第三章第二節, 及第三章第三節。同時提供一些相關例題, 輔以闡述概念。例題取自黃文璋(2003a)第三章習題。

詳細內容可參閱下列網站http://www.math.nuk.edu.tw/cbme/cbme.htm。

設X, Y為二離散型的r.v's, 則對任意R之一可數的子集B, 只要$P(X = x) > 0$, 則給定$X = x, Y$之條件機率為

$$P(Y \in B|X = x) = \sum_{y \in B} P(Y = y|X = x) = \sum_{y \in B} h(y|x), \tag{1}$$

其中$h(y|x)$為給定$X = x, Y$之條件p.d.f.。若X, Y為連續型的r.v.'s, 因$P(X = x) = 0, \forall x \in R$, 故此時給定$X = x, Y \in B$之條件機率本來是沒有定義的。但如同(1)式, 只要$g(x) > 0$, 其中g為X之p.d.f., 對$\forall B \subset R$, 我們定義

$$P(Y \in B|X = x) = \int_B h(y|x)dy, \tag{2}$$

只要上式右側積分存在。而條件期望值之定義則如下所述。設有X, Y二r.v.'s, ω為一Borel函數, 令$Z = \omega(X, Y)$。先設X, Y為二離散型之r.v.'s, 以f為其聯合p.d.f., 且令g, h分別表X及Y之邊際p.d.f's。又令$D = \{x|x \in R, g(x) > 0\}, K = \{y|y \in R, h(x) > 0\}$。對$\forall x \in D$, 定義給定$X = x$, Z之條件期望值為

$$E(Z|X = x) = \sum_{y \in K} \omega(x, y)h(y|x), \tag{3}$$

只要右側和絕對收斂。此處$h(y|x)$表給定$X = x, Y$之條件p.d.f.。若X, Y為連續型, 則對$\forall x \in D$, 定義給定$X = x, Z$之條件期望值為

$$E(Z|X = x) = \int_{-\infty}^{\infty} \omega(x, y)h(y|x)dy, \tag{4}$$

只要右側積分絕對收斂。此處$h(y|x)$亦為給定$X = x, Y$之條件p.d.f.。底下我們給出幾個關於條件機率與條件期望值的例子, 透過例子的計算可以較清楚的了解條件機率與條件期望值。

例1. 設給定$X=x, Y$有$\mathcal{N}(x,\tau^2)$分佈，又X有$\mathcal{N}(\mu,\sigma^2)$分佈。試證$Y$有$\mathcal{N}(\mu,\sigma^2+\tau^2)$分佈。

解：$Y|X=x \sim \mathcal{N}(x,\tau^2),\ X \sim \mathcal{N}(\mu,\sigma^2)$,

$$\begin{aligned} f_{X,Y}(x,y) &= f_{Y|X}(y|x) f_X(x) \\ &= \frac{1}{\sqrt{2\pi}\tau}\ e^{-\frac{(y-x)^2}{2\tau^2}} \frac{1}{\sqrt{2\pi}\sigma}\ e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ &= \frac{1}{2\pi\tau\sigma}\ e^{-\frac{\sigma^2(y-x)^2+\tau^2(x-\mu)^2}{2\tau^2\sigma^2}} \end{aligned}$$

則Y的分佈為

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\tau\sigma}\ e^{-\frac{\sigma^2(y-x)^2+\tau^2(x-\mu)^2}{2\tau^2\sigma^2}} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\tau\sigma}\ e^{\frac{\sigma^2+\tau^2}{2\sigma^2\tau^2}\left\{[x-\frac{\sigma^2 y+\tau^2\mu}{\sigma^2+\tau^2}]^2+\frac{\sigma^2\tau^2}{(\sigma^2+\tau^2)^2}(y-\mu)^2\right\}} dx \\ &= \frac{1}{\sqrt{2\pi}\sqrt{(\sigma^2+\tau^2)}}\ e^{-\frac{1}{\sigma^2+\tau^2}(y-\mu)^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{\tau\sigma}{\sqrt{\sigma^2+\tau^2}}}\ e^{\frac{1}{2\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}}[x-\frac{\sigma^2 y+\tau^2\mu}{\sigma^2+\tau^2}]^2} dx \\ &= \frac{1}{\sqrt{2\pi}\sqrt{(\sigma^2+\tau^2)}}\ e^{-\frac{1}{\sigma^2+\tau^2}(y-\mu)^2} \end{aligned}$$

故$Y \sim \mathcal{N}(\mu,\sigma^2+\tau^2)$。

例2. 設X有$\mathcal{U}(0,1)$分佈，且給定$X=x, 0<x<1, Y$有$\mathcal{G}e(x)$分佈。試求$P(X>\frac{1}{2}|Y=y), y=0,1,\cdots$。

解：$Y|X=x \sim \mathcal{G}e(x),\ X \sim \mathcal{U}(0,1)$,

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = x(1-x)^{y-1}$$

因此

$$f_Y(y) = \int_0^1 x(1-x)^{y-1} dx = \frac{1}{y^2+y}$$

則

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = x(1-x)^{y-1}(y^2+y),$$

故

$$
\begin{aligned}
P(X > \frac{1}{2}|Y=y) &= \int_{1/2}^{1} \frac{f_{X,Y}(x,y)}{f_Y(y)} = x(1-x)^{y-1}(y^2+y) \ (\text{令} z = 1-x) \\
&= \int_0^{1/2} (1-z)z^{y-1}(y^2+y)dz \\
&= (y^2+y)(\frac{1}{y}z^y - \frac{1}{y+1}z^{y+1})|_0^{1/2} \\
&= \frac{2+y}{2^{y+1}}
\end{aligned}
$$

例3. 設X, Y之聯合p.d.f.為$f(x,y) = k(k-1)(y-x)^{k-2}, 0 < x \leq y < 1, k \geq 2$為一整數。

(i) 試求$E(X|Y)$;

(ii) 試利用(i)求$E(E(X|Y))$。

解：$X, Y \sim f(x,y) = k(k-1)(y-x)^{k-2}, 0 < x \leq y < 1, k \geq 2$, 則

$$f_Y(y) = \int_0^y f(x,y)dx = \int_0^y k(k-1)(y-x)^{k-2}dx = ky^{k-1}$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = (k-1)\frac{1}{y}(1-\frac{x}{y})^{k-2}$$

(i)

$$E(X|Y) = \int_0^y x f_{X|Y}(x|y)dx = \int_0^y \frac{k-1}{y^{k-1}} x(y-x)^{k-2}dx = \frac{y}{k}$$

(ii)

$$E(E(X|Y)) = E(\frac{Y}{k}) = \int_0^1 \frac{y}{k} f_Y(y)dy = \int_0^1 \frac{y}{k} ky^{k-1}dy = \frac{1}{1+k}$$

例4. 設X, Y之聯合p.d.f.為$f(x,y) = x+y, 0 < x, y < 1$。試求

(i) $E(X|Y=y)$;

(ii) $E(Xe^{Y+Y^{-1}}|Y=y)$。

解：$X, Y \sim f(x,y) = x+y, 0 < x, y < 1$, 則

$$f_Y(y) = \int_0^1 (x+y)dx = \frac{1}{2} + y,$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x+y}{\frac{1}{2}+y} = \frac{2x+2y}{1+2y}.$$

(i)

$$E(X|Y=y) = \int_0^1 x\frac{2x+2y}{1+2y}dx = \frac{2}{1+2y}\int_0^1 (x^2+xy)dx = \frac{2+3y}{3(1+2y)}$$

(ii)

$$E(Xe^{Y+Y^{-1}}|Y=y) = e^{y+y^{-1}}E(X|Y=y) = \frac{e^{y+y^{-1}}(2+3y)}{3(1+2y)}$$

參考資料

1. 黃文璋(2003a). 機率論。華泰文化事業公司，台北。

2. 黃文璋(2003b). 數理統計。華泰文化事業公司，台北。