國立中山大學 104 學年度寒假轉學考招生考試試題

科目名稱：線性代數【應數系二年級】 題號：20402

※本科目依簡章規定「不可以」使用計算機 共 1 頁 第 1 頁

1. [15%] Let

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & a \end{pmatrix},$$

where $a, \theta \in \mathbb{R}$.

   (1) Find A^n, $n \in \mathbb{N}$.

   (2) Show that A is invertible if and only if $a \neq 0$. Find A^n, $n \in \mathbb{Z}$, if A is invertible.

2. [15%] Find the Jordan canonical form of the matrix

$$\begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}.$$

3. [15%] Let V be a vector space, and let $T : V \to V$ be a linear transformation satisfying $T^n = I$, the identity transformation, for some $n \in \mathbb{N}$. Find the range and kernel of T.

4. [15%] Let A, B, and $A + B$ be invertible $n \times n$ matrices. Show that $A^{-1} + B^{-1}$ is invertible. Find the inverse of $A^{-1} + B^{-1}$ (in terms of A, B, $A + B$ and their inverses).

5. [20%] Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Show that one and exactly one of the following situations (1) and (2) occurs:

   (1) $T(v) = 0$ for some non-zero vector $v \in V$;

   (2) $T(x) = u$ has a solution x in V for every $u \in V$.

6. [20%] Let V be a vector space with $\dim V = n$. Suppose that $T : V \to V$ is a linear transformation and $v \in V$ satisfying $A^{n-1}v \neq 0$ and $A^n v = 0$. Show that A admits the matrix representation

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

with respect to some basis of V.