國立中山大學 106 學年度寒假轉學考招生考試試題

科目名稱：線性代數【應數系二年級】 題號：20402

※本科目依簡章規定「不可以」使用計算機 共 1 頁第 1 頁

1. [10%] Find all values of k for which the set $\{(1,3,1),(-2,4,5),(1,1,k)\}$ of vectors in $\mathbb{R}^3$ is linearly dependent.

2. [15%] Let $a_1,\ldots,a_n$ be numbers in the complex field $\mathbb{C}$. Determine whether the matrix

$$\begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with 0's on the diagonal entries and $a_1,\ldots,a_{n-1}$ on the upper off-diagonal entries is invertible. If the matrix is invertible, find its inverse.

3. [15%] Show that the set $\mathbb{C}[X]$ of all polynomial over $\mathbb{C}$ is a vector space. Find the dimension of the subspace of $\mathbb{C}[X]$ spanned by the set

$$\{1,(X-1),(X-2)^2,\ldots,(X-n)^n\},$$

and verify your assertion.

4. [15%] Find the Jordan canonical form of the matrix

$$\begin{pmatrix} 3 & 0 & 8 \\ 3 & -1 & 6 \\ -2 & 0 & -5 \end{pmatrix}.$$

5. [15%] Denote by $\mathrm{Tr}_n : M_n(\mathbb{C}) \to \mathbb{C}$ the trace map on the space of $n\times n$ matrices over $\mathbb{C}$, i.e.,

$$\mathrm{Tr}_n A = \sum_{i=1}^{n} A_{ii}$$

for any $A=(A_{ij})\in M_n(\mathbb{C})$. Show that

$$\mathrm{Tr}_n(AB) = \mathrm{Tr}_m(BA)$$

for any matrices $A\in M_{n\times m}(\mathbb{C})$ and $B\in M_{m\times n}(\mathbb{C})$.

6. [30%] For any $n\in\mathbb{N}$, denote by $P_n(\mathbb{R})$ the set of polynomials over $\mathbb{R}$ with degrees less or equal to n. Define a map $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ by

$$T(p(x)) = (x+1)p'(x) + \int_{-x}^{x} p(t)\,dt,$$

where $'$ is the first-order derivative.

   (1) Prove that T is linear.

   (2) Find the matrix representation of T with respect to the ordered basis $\{1,x,x^2\}$ and $\{1,x,x^2,x^3\}$ for $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$, respectively.